

# Exact bearing capacity calculations using the method of characteristics

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**Keywords:** bearing capacity, shallow foundation, cohesive-frictional, limit analysis

**ABSTRACT:** This paper discusses the use of the method of characteristics (commonly referred to as the slip-line method) to solve the classic geotechnical bearing capacity problem of a vertically loaded, rigid strip footing resting on a cohesive-frictional halfspace. It would appear that, contrary to popular belief, the method of characteristics can be used to establish the exact plastic collapse load for any combination of the parameters  $c$ ,  $\phi$ ,  $\gamma$ ,  $B$  and  $q$  – including the infamous ‘ $N_\gamma$  problem’. This applies to footings of arbitrary roughness, though only the extreme cases (smooth and fully rough) are considered in detail here.

## 1 Introduction

Many analytical and numerical techniques can be used to calculate the vertical bearing capacity of a rigid strip footing. These include the method of characteristics, upper bound and limit equilibrium calculations based on assumed mechanisms, finite element formulations of the lower and upper bound theorems, and conventional finite element or finite difference analyses employing displacement elements. (Note that here and throughout the paper, all references to lower bound, upper bound and exact plasticity solutions imply the assumption of an associated flow rule.) If the self-weight of the soil is neglected, the method of characteristics is the preferred technique because it allows the construction of a simple lower bound stress field from which the bearing capacity can be determined in closed algebraic form. The resulting expressions for  $N_c$  and  $N_q$  are universally adopted in design when applying Terzaghi’s equation:

$$q_u = Q_u / B = cN_c + qN_q + \frac{1}{2}\gamma BN_\gamma \quad (1)$$

There are several ways of showing that the stress fields for  $N_c$  and  $N_q$  (which have identical meshes of characteristics) can be extended throughout the semi-infinite soil mass without violating equilibrium or yield. For weightless soil it is also straightforward to derive a rigorous upper bound on the bearing capacity that coincides precisely with the lower bound, and furthermore it can be shown that the roughness of the footing has no effect on the collapse load. When the soil has self-weight, however, the status of (numerical) solutions obtained using the method of characteristics is far less certain; there is also debate over how footings of different roughness should be handled. These issues are frequently used as a justification for adopting the other calculation techniques mentioned above. Indeed, it would be fair to say that the method of characteristics has acquired something of a credibility problem in recent years. The view of Frydman & Burd (1997) is typical: “Evaluation of the third coefficient,  $N_\gamma$ , requires that the self-weight of the soil be considered. In this case, stress characteristics fields may be obtained numerically, but it is not generally clear whether the resulting slip mechanisms (which include curved slip lines) are also kinematically admissible, or whether the stress field can be extended outside the identified plastic zone. Consequently, it is not clear that the solutions obtained using this technique ... are exact, or even lower bounds to the

exact solution ... A further complication exists in the use of the method of characteristics when the base of the footing is assumed to be fully rough. In this case ... the precise nature of the boundary conditions that should be applied at the base of the footing is not clear". Similar observations have been made by a number of other authors (see e.g. Ukritchon et al., 2004; Hjiiaj et al., 2005).

It is true that, when the soil has self-weight, previous studies of bearing capacity using the method of characteristics have often lacked theoretical rigour. Sometimes this has been freely acknowledged: Cox (1962) admits that his analyses "are concerned solely with the calculation of a region of the stress field sufficiently extensive to determine the stresses on the indenter, it being assumed that the rigorous mathematical justification of the results obtained is merely a matter of additional computation", while Salençon & Matar (1982) acknowledge that "To be quite rigorous ... it would be necessary to complete the stress field in the entire soil layer ... and to build a velocity field associated to this stress field by using the mathematic rule of normality ... for reason of simplicity this question will be left aside here and afterwards". In many other studies, the important questions of stress field extensibility and kinematic admissibility have only been mentioned in passing, if at all (Lundgren & Mortensen, 1953; Sokolovskii, 1965; Larkin, 1968 and discussers; Graham & Stuart, 1971; Ko & Scott, 1973; Bolton & Lau, 1993; Kumar, 2003). The paper by Davis & Booker (1971) is a notable exception in that both issues are acknowledged, and backed up with appropriate calculations (or at least reports thereof). Although their work represents perhaps the most rigorous treatment of the strip footing problem to date, it is unsatisfactory in several respects. First, they concentrate on the problem of a fully rough footing, on the grounds that "adaptation of the analysis to the smooth case is relatively straightforward". They also appear to be under the impression that the smooth footing solutions of Cox (1962) are rigorous, when in fact they are not (see above). Second, their discussion and illustration of the velocity field is somewhat obscured by the consideration of a general non-associated flow rule,  $\psi \leq \phi$ . Third, although they claim "In all cases it was then found that the stresses satisfied equilibrium and did not exceed the strength of the material", a completed stress field is only depicted for one (unspecified) case when  $\phi = 30^\circ$ . Fourth, they simply declare that "the velocity field was evaluated for each problem and the rate of plastic work calculated at the nodes of the characteristic net. In every case this rate was nowhere negative thus confirming the kinematic validity of the solutions". No further details of these calculations are given, and no actual upper bounds are evaluated, so their claims to have established exactness are not totally convincing.

The other major area of concern relates to the boundary conditions that should be applied when using the method of characteristics to solve problems involving rough (or semi-rough) footings. It is certainly true that there is confusion in the literature, dating back to Caquot & Kerisel (1953). Other questionable analyses of rough footings have been made by Graham & Stuart (1971), Bolton & Lau (1993) and Kumar (2003). These authors make various *a priori* assumptions about the stress field under the footing, often involving a wedge of non-plastic soil having some predetermined shape. Such assumptions are unnecessary, and invariably lead to erroneous results when the soil has self-weight. In fact, the shape of this wedge or 'false head', which may or may not span the full width of the footing, emerges naturally during the iterative process needed to determine the extent of the mesh of characteristics for a particular combination of parameters. The correct procedure has been outlined (and used) by Lundgren & Mortensen (1953) and Davis & Booker (1971), though admittedly the clarity of their explanations and figures could have been better.

Within the limited space available, this paper attempts to address some key questions concerning bearing capacity calculations using the method of characteristics:

- Is the collapse load obtained from the incomplete stress field exact, even when  $\gamma > 0$ ?
- What makes the  $N_\gamma$  problem so difficult to solve? Can exact values of  $N_\gamma$  be obtained?

As mentioned in the Abstract, only the extreme cases of smooth and fully rough footings will be considered in the example problems. Some values of  $N_\gamma$  for semi-rough footings will be given, but their derivation will not be discussed here (generally speaking, intermediate roughness is no more complicated than full roughness). All analyses are restricted to plane strain geometry and are based on the classical assumptions of rigid-perfectly plastic soil behaviour, i.e. a linear Mohr-Coulomb yield envelope with an associated flow rule ( $\psi = \phi$ ). The latter assumption means that the stress and velocity characteristics coincide; they indicate the planes on which the Mohr's circle of stress touches the yield envelope, and the directions in which the extensional strain is zero.

## 2 Example problems

Stress and velocity calculations for vertically loaded footings using the method of characteristics have been implemented in the publicly available computer program ABC (Martin, 2004). The user manual gives full details of the relevant theory, the numerical methods employed, and the extensive collections of test problems that have been used for validation. The examples in Table 1 will be used to illustrate the process of establishing the exact bearing capacity. All three problems have been studied before – the first by Cox (1962) and the other two by Salençon & Matar (1982) – but these previous analyses were confined to calculation of the stress field (without completion).

Figure 1 shows the stress and velocity fields for the three problems, using coarse meshes of characteristics for clarity. In Figure 1(a) the footing is smooth, so every  $\alpha$  characteristic proceeds from the soil surface to the underside of the footing. An iterative process is used to determine the extent of the mesh, such that the final (i.e. outermost)  $\alpha$  characteristic intersects the footing on the axis of symmetry. In the associated or 'consistent' velocity field, there are discontinuities along the underside of the footing and along the outside of the mesh (the exterior soil being assumed rigid). In Figure 1(b) the footing is fully rough, but for this combination of parameters, the need to satisfy the symmetry condition on the  $z$  axis means that no  $\alpha$  characteristics can proceed to the footing. Instead they all terminate in mid-soil, and there is never any need to apply the rough footing boundary condition. The iterative adjustment process now involves two variables, namely, the extent of the mesh along the soil surface, and the aperture of the fan. In the consistent velocity field, the blank false head region moves down with the footing as a rigid body, giving rise to a velocity discontinuity along the innermost  $\beta$  characteristic. In Figure 1(c) the footing is again rough, but now the problem parameters are such that the inner  $\alpha$  characteristics can proceed to the footing. The  $\beta$  characteristics that emerge from the endpoints are tangential to the footing, and on this part of the interface the full shear strength of the soil is mobilised. The remaining  $\alpha$  characteristics terminate in mid-soil, resembling those of the second example. Construction of this type of mesh requires the iterative adjustment of two distances along the soil surface, delineating the  $\alpha$  characteristics that proceed to the footing from those that do not. In the consistent velocity field the blank false head region again moves down with the footing; there are velocity jumps along the innermost  $\beta$  characteristic and along the part of the footing where full roughness is mobilised. Having constructed the stress and velocity fields, two separate calculations of the bearing capacity can be performed: a 'stress calculation' involving integration of the boundary tractions under the footing (see Figure 1), and a 'velocity calculation' involving evaluation of the internal and external work rates. It is important to realise that these two calculations are approximate – they incorporate errors introduced by the use of finite difference (rather than exact) equations when constructing the stress and velocity fields, as well as errors resulting from the use of numerical integration in the calculations themselves. It is of course possible to perform a sequence of increasingly accurate analyses, each one involving a finer mesh of characteristics, until the results converge to the desired precision. Until this has been done, however, it is not appropriate to refer to the stress and velocity calculations as lower and upper bound calculations (e.g. as a mesh is refined, the result of the 'upper bound' calculation may converge from below). Table 2 shows the results obtained. Despite the coarseness of the initial meshes, the stress and velocity calculations for each problem converge very quickly as the subdivision counts are doubled, quadrupled, etc. The original results reported by Cox (1962) and Salençon & Matar (1982) were 37.8,  $1.61 \times 10^3$  and 44.9 kPa, so the level of agreement is encouraging. Even more encouraging is the fact that, in all three problems, the stress and velocity calculations converge to identical values.

Table 1. Example problems.

Problem no.	$c$ [kPa]	$\phi$ [°]	$\gamma$ [kN/m <sup>3</sup> ]	Footing type	$B$ [m]	$q$ [kPa]
1	1	20	10	Smooth	2	0
2	16	30	18	Rough	4	18
3	$1 + 2.5z$	10	16	Rough	4	0

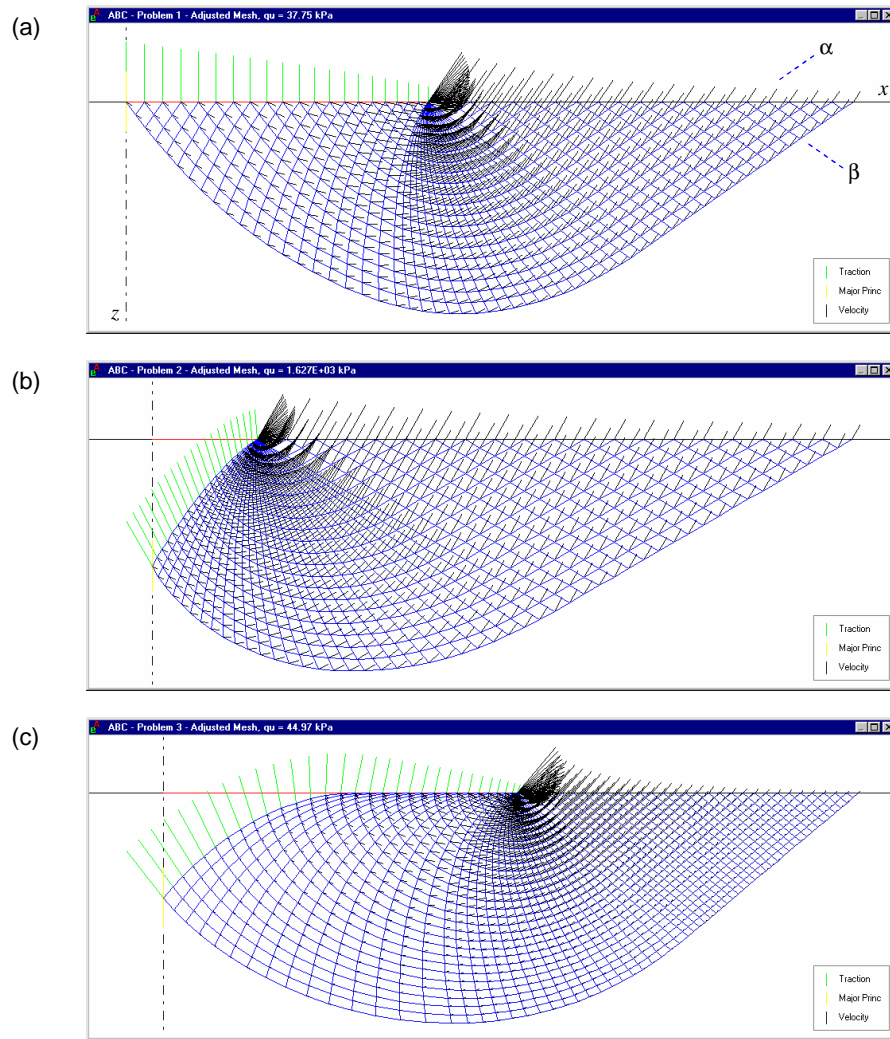


Figure 1. Stress and velocity fields for example problems.

Table 2. Convergence of  $q_u$  (kPa) in example problems.

Mesh	Problem 1 stress calc.	Problem 1 velocity calc.	Problem 2 stress calc.	Problem 2 velocity calc.	Problem 3 stress calc.	Problem 3 velocity calc.
Initial (Fig. 1)	37.75	37.77	$1.627 \times 10^3$	$1.627 \times 10^3$	44.97	44.99
$\times 2$	37.76	37.77	$1.626 \times 10^3$	$1.626 \times 10^3$	44.99	44.99
$\times 4$	37.76	37.76	$1.626 \times 10^3$	$1.626 \times 10^3$	44.99	44.99
$\times 8$	37.76	37.76	$1.626 \times 10^3$	$1.626 \times 10^3$	44.99	44.99
$\times 16$	37.76	37.76	$1.626 \times 10^3$	$1.626 \times 10^3$	44.99	44.99
$\times 32$	37.76	37.76	$1.626 \times 10^3$	$1.626 \times 10^3$	44.99	44.99

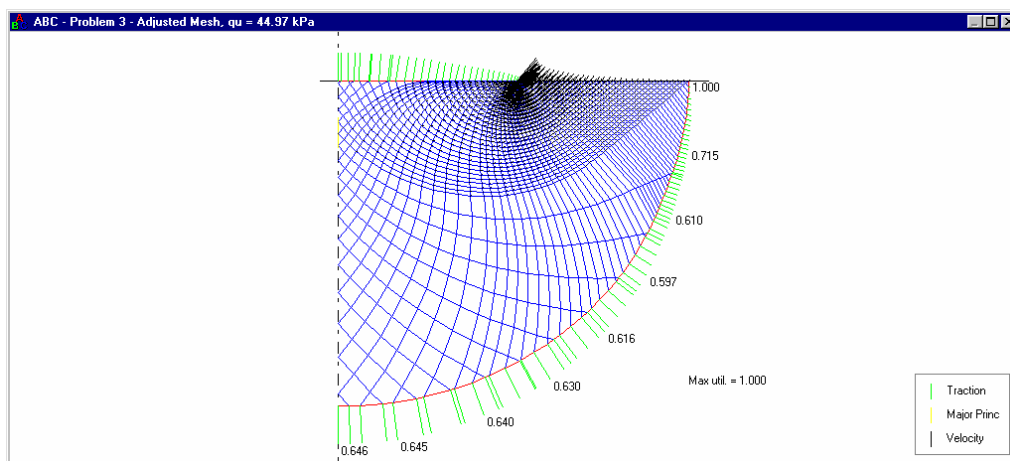
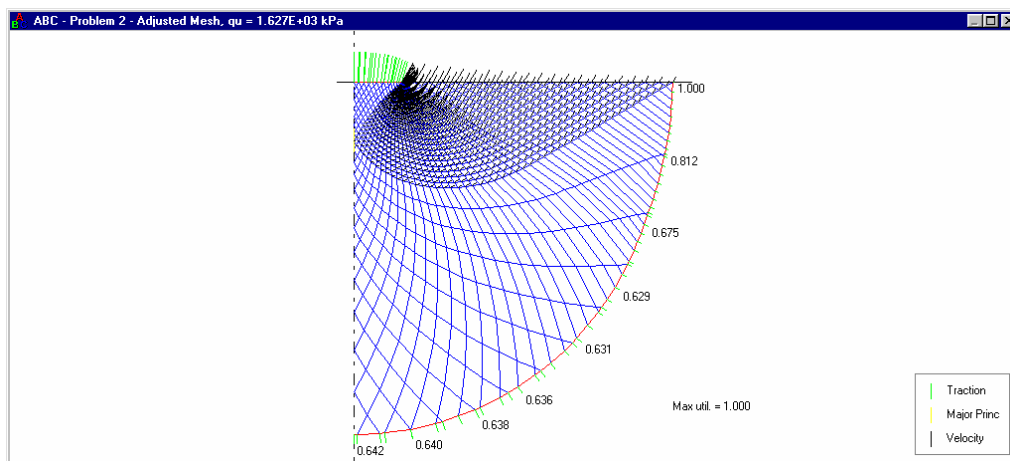
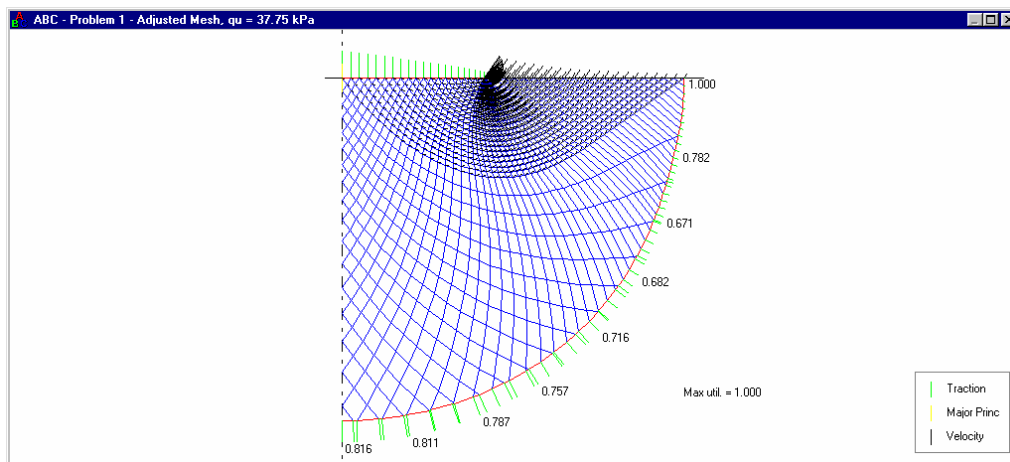


Figure 2. Completed stress fields for example problems.

Once converged, the velocity calculations in Table 2 constitute strict upper bound solutions. To confirm that the converged stress calculations are strict lower bounds (thereby establishing exactness) the stress fields of Figure 1 must be completed. This can be done by following Davis & Booker (1971) and using an ingenious extension strategy originally developed by Cox et al. (1961) for weightless soil. The basic idea, as shown in Figure 2, is to extend the stress field downwards and outwards by 'bouncing' characteristics off the  $z$  axis, continuing as far as the minor principal stress trajectory originating from the extremity of the original mesh. The stresses in the extension need not be fully plastic (except at the boundary with the original mesh), but this is a convenient assumption since the governing equations and finite difference routines then remain unchanged. On the other side of the minor principal stress trajectory, the stress field is comprised of columns or 'spokes' of uniaxial stress, superimposed on a hydrostatic stress field. To be more precise, if the principal stresses (compression positive) at a point  $(x_0, z_0)$  on the interior of the trajectory are  $\sigma_1$  and  $\sigma_3$ , then the principal stresses at any point  $(x, z)$  along the supporting spoke are given by

$$\sigma_{1,\text{ext}} = \sigma_1 + \gamma(z - z_0) ; \sigma_{3,\text{ext}} = \gamma z + q \quad (2)$$

In the limiting case of an infinite number of spokes, the stress field outside the minor principal stress trajectory satisfies equilibrium (Cox et al. 1961). It is also easy to show that in a given spoke, the yield criterion will be satisfied everywhere provided it is satisfied at the starting point  $(x_0, z_0)$ . The verification of the extension therefore reduces to a check on the yield criterion at the start of each spoke. For this it is convenient to define a utilisation factor that compares the size of the Mohr's circle of stress with the maximum allowable size at the same level of mean stress:

$$\text{Utilisation} = \frac{\sigma_1 - \gamma z_0 - q}{2c \cos \phi + (\sigma_1 + \gamma z_0 + q) \sin \phi} \quad (3)$$

If all the utilisation factors are less than or equal to unity, the extension is safe, and the bearing capacity calculated from the incomplete stress field is confirmed as a strict lower bound. Note that the yield criterion is checked at all points on the trajectory, not just those where labels are printed in Figure 2. For rough footings it is also necessary to extend the stress field into the false head region, whether it spans all or part of the footing. This is a straightforward operation that involves bouncing characteristics off the  $z$  axis, followed by truncation of the stress field along the underside of the footing. The completed stress fields in Figure 2 are all statically admissible, and remain so during refinement, so the converged results in Table 2 are formally established as exact.

### 3 Evaluation of $N_g$

The bearing capacity factor  $N_\gamma$  depends on the friction angle of the soil and the roughness of the footing (it also depends on the dilation angle of the soil, but the aim here is to obtain definitive solutions for the case  $\psi = \phi$ ). Many graphs and tables of recommended  $N_\gamma$  values have been published, but the range covered by these solutions is large, especially for high friction angles and rough footings. It is clearly desirable to resolve this issue, at least for the case of an associated flow rule, when the uniqueness theorem guarantees that there is an unequivocal, exact value of  $N_\gamma$  for each combination of friction angle and roughness. While there is no need for the bearing capacity factors used in design to be particularly accurate, if the exact values of  $N_\gamma$  can be found, then it would surely make sense to use them (just as the exact values of  $N_c$  and  $N_q$  are used). When employing the method of characteristics, it is numerically convenient (see e.g. Larkin, 1968; Graham & Stuart, 1971; Davis & Booker, 1971; Bolton & Lau, 1993; Martin, 2004) to consider the  $N_\gamma$  problem as a limiting case of bearing capacity on cohesionless soil:

$$\left. \begin{aligned} N_q &= \lim_{\gamma B/q \rightarrow 0} q_u/q = \lim_{\gamma B/q \rightarrow 0} Q_u/qB \\ N_\gamma &= \lim_{\gamma B/q \rightarrow \infty} 2q_u/\gamma B = \lim_{\gamma B/q \rightarrow \infty} 2Q_u/\gamma B^2 \end{aligned} \right\} \text{with } c = 0 \quad (4)$$

It is also possible to consider a cohesive-frictional soil with no surcharge, letting the cohesion tend to zero, though the approach in equation (4) is usually preferred because of its transparent physical interpretation ( $q/\gamma B$  corresponds directly to the familiar embedment ratio  $D/B$ ).

Table 3. Determining  $N_\gamma$  from analyses in which  $c = 0$  and  $\gamma B/q \rightarrow \infty$  (see equation (4)).

$\gamma B/q$	$\phi = 10^\circ$ Smooth	$\phi = 10^\circ$ Rough	$\phi = 30^\circ$ Smooth	$\phi = 30^\circ$ Rough	$\phi = 50^\circ$ Smooth	$\phi = 50^\circ$ Rough
$10^3$	0.2938	0.4495	7.780	14.91	374.4	745.5
$10^4$	0.2826	0.4354	7.670	14.77	372.3	743.2
$10^5$	0.2811	0.4334	7.655	14.76	372.0	742.9
$10^6$	0.2809	0.4332	7.653	14.75	372.0	742.9
$10^9$	0.2809	0.4332	7.653	14.75	372.0	742.9
$10^{12}$	0.2809	0.4332	7.653	14.75	372.0	742.9

If highly accurate values of  $N_\gamma$  are sought – obviously anything more than 2-digit precision is purely for academic interest – then  $\gamma B/q$  needs to be surprisingly large. For example, Table 3 shows that  $\gamma B/q$  must be at least  $10^6$  in order to obtain  $N_\gamma$  factors that are correct to 4 digits over the range  $\phi = 10^\circ$  to  $50^\circ$ . Note that these results are specific to program ABC, which uses a particular adaptive subdivision strategy to maintain the accuracy of the stress field calculation near the edge of the footing. This can be seen in Figure 3 (drawn for  $\gamma B/q = 10^9$ ) where the light-coloured characteristics are those resulting from an adaptive subdivision. Many researchers have noted that as  $\gamma B/q$  becomes large, the usual fan zone becomes severely distorted, and degenerates into a single  $\beta$  characteristic as  $\gamma B/q \rightarrow \infty$  (see arrows in Figure 3). For accurate results, an extremely fine mesh is needed in the vicinity of the footing edge, and adaptive subdivision (Larkin, 1968; Graham & Stuart, 1971; Martin, 2004) can achieve this far more efficiently than a fine but uniform spacing (Bolton & Lau, 1993; Kumar, 2003). Another peculiarity of the  $N_\gamma$  problem is the consistent velocity field: as  $\gamma B/q \rightarrow \infty$ , the velocities emerging close to the edge of the footing also approach infinity. If the entire velocity field is plotted to scale, as in Figure 3, the picture is not very informative.

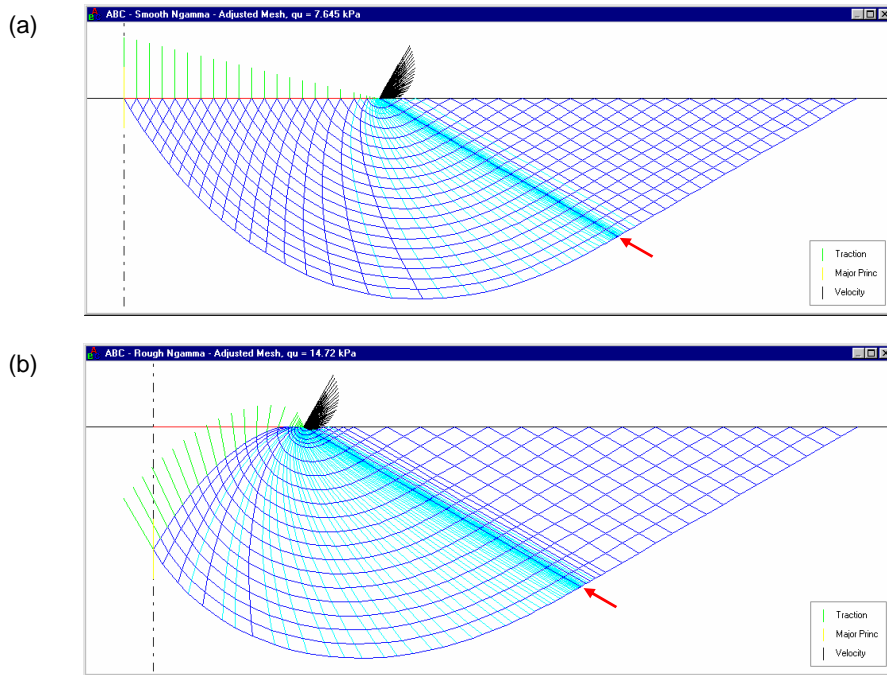


Figure 3. Stress and velocity fields for  $N_\gamma$  problem ( $c = 0$ ,  $\phi = 30^\circ$ ,  $\gamma B/q = 10^9$ , smooth and rough).

Each bearing capacity in Table 3 is a converged solution obtained by systematic refinement of the relevant mesh of characteristics. Each value is also exact for its combination of  $\gamma B/q$ ,  $\phi$  and roughness (in all cases it is found that the converged stress and velocity calculations agree, and the stress field is extensible). For the case  $\phi = 30^\circ$  and  $\gamma B/q = 10^9$ , details of the convergence behaviour are given in Table 4. The stress results converge quickly, but the velocity calculations now require several refinements before the desired 4-digit precision is achieved (cf. Table 2). In view of the comments on the velocity field in the previous paragraph, this is not surprising. Having determined that the converged stress and velocity calculations agree, completion of the stress field is the final formality needed to confirm exactness. This is illustrated in Figure 4, again for the case  $\phi = 30^\circ$  and  $\gamma B/q = 10^9$ . The utilisations are acceptable, and remain so with further refinement.

The parametric study in Table 3 suggests that, when using the approach of equation (4), a  $\gamma B/q$  ratio of  $10^9$  is comfortably large enough to evaluate  $N_\gamma$  to 4-digit precision. This has been confirmed for friction angles in the range  $1^\circ$  to  $60^\circ$ , for both smooth and rough footings. A selection of  $N_\gamma$  values is given in Table 5, and these have all been confirmed as exact solutions in the manner of Table 4 and Figure 4 (readers wishing to verify this are welcome to download ABC and run it with the appropriate parameters). A fuller version of Table 5, covering the range  $\phi = 1^\circ$  to  $60^\circ$  in  $1^\circ$  intervals and giving  $N_\gamma$  to 6-digit precision, will appear in a forthcoming journal paper; it can also be downloaded from the author's website in spreadsheet form.

It is worth pointing out that numerical values of  $N_\gamma$  for any friction angle and roughness can be obtained much more quickly without using the method of characteristics. When  $c = q = 0$  exactly, the equations governing the stress field reduce to a pair of ordinary differential equations that can be solved by Runge-Kutta or similar techniques (Lundgren & Mortensen, 1953; Sokolovskii, 1965; Booker, 1970; Salençon, 1977). This approach allows an  $N_\gamma$  factor to be calculated to very high precision in a fraction of the time that it takes to construct (and then systematically refine) a mesh of characteristics for a large value of  $\gamma B/q$ . It is therefore ideal for compiling a large table of values, and the calculations for the above-mentioned spreadsheet were performed in this way. Although it gives the same result for  $N_\gamma$  as taking the limit in equation (4), the drawback of the alternative approach is that it effectively amounts to the calculation of an incomplete stress field. If proof of exactness is required, it is necessary to revert to the method of characteristics, at least to construct the velocity field (it may well be possible to complete the stress field by solving ODEs only).

Table 4. Convergence in  $N_\gamma$  problem ( $c = 0$ ,  $\phi = 30^\circ$ ,  $\gamma B/q = 10^9$ ).

Mesh	Smooth stress calc.	Smooth velocity calc.	Rough stress calc.	Rough velocity calc.
Initial (Fig. 3)	7.645	7.682	14.72	14.82
× 2	7.651	7.660	14.75	14.77
× 4	7.653	7.655	14.75	14.76
× 8	7.653	7.653	14.75	14.76
× 16	7.653	7.653	14.75	14.75
× 32	7.653	7.653	14.75	14.75

Table 5. Selected exact values of  $N_\gamma$ .

$\phi$ [°]	Smooth	$\delta/\phi = 1/3$	$\delta/\phi = 1/2$	$\delta/\phi = 2/3$	Rough
10	0.2809	0.3404	0.3678	0.3929	0.4332
20	1.579	2.167	2.411	2.606	2.839
30	7.653	11.75	13.14	14.03	14.75
40	43.19	73.55	80.62	83.89	85.57
50	372.0	690.8	728.9	739.8	742.9



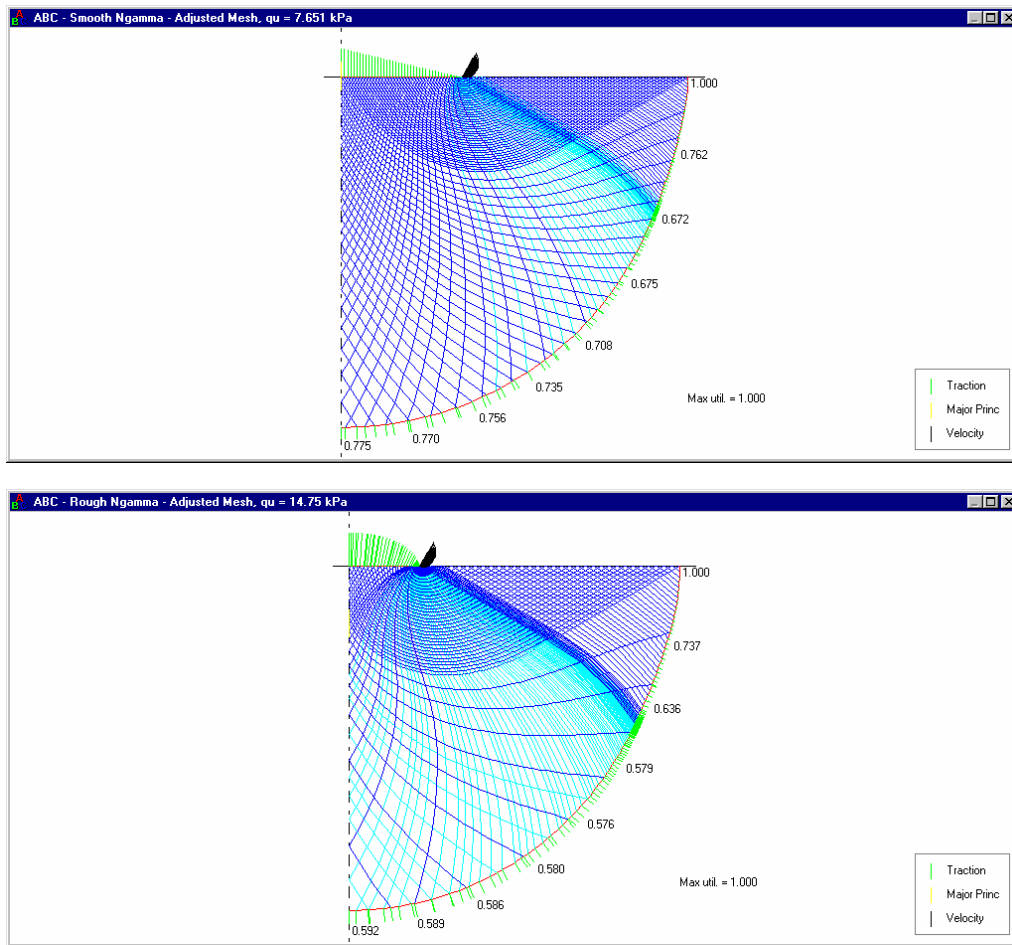


Figure 4. Completed stress fields for  $N_\gamma$  problem ( $c = 0$ ,  $\phi = 30^\circ$ ,  $\gamma B/q = 10^9$ , smooth and rough).

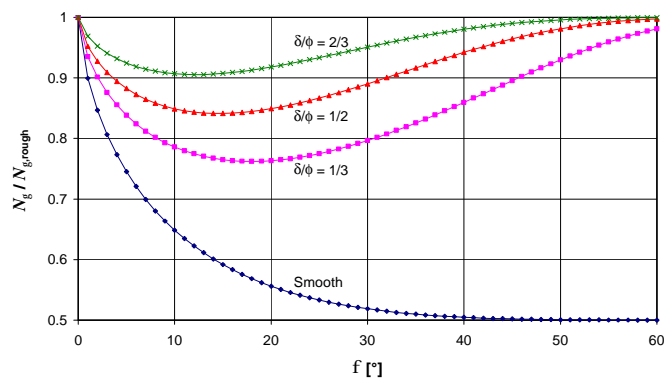


Figure 5. Influence of roughness on  $N_\gamma$ .

Figure 5 shows the effect of roughness on  $N_f$ . Rather surprisingly, footings with interface friction angles as small as  $\phi/3$  still provide at least 75% of the fully rough capacity, regardless of  $\phi$ . The smooth footing also exhibits intriguing behaviour – at large friction angles, its bearing capacity approaches exactly half of the rough capacity. The mathematical reason for this is by no means obvious, but the circumstantial evidence is compelling (the capacity ratio for  $60^\circ$  is 0.500043). One final remark on the  $N_f$  problem: for rough or semi-rough footings, the false head never spans the full width of the footing, i.e. the solutions are always of the type shown in Figures 1(c) and 3(b). Since the fan is degenerate, it is obviously impossible to have a solution of the type in Figure 1(b).

## 4 Conclusions

The method of characteristics is not a versatile technique, but the strip footing bearing capacity problem is one that it can solve, and solve exactly when  $\psi = \phi$ . Apart from the examples presented here, several hundred cases covering the full spectrum of soil and footing parameters have been analysed, and the results obtained have always been confirmed as exact plasticity solutions (though sometimes a small modification to the basic stress field extension strategy is required). Proponents of other techniques would do well to examine the accuracy of their own solutions for this fundamental benchmark problem, especially with regard to the bearing capacity factor  $N_f$  (e.g. the rough footing results of Frydman & Burd (1997) are all high by 40 to 80%, and about one-third of the supposedly rigorous lower bounds of Hjiatj et al. (2005) lie slightly above the exact values).

## 5 Acknowledgements

This paper was written while the author was visiting the Centre for Offshore Foundation Systems in Perth. The financial support arranged by Prof. Mark Randolph is gratefully acknowledged.

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